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LETTER TO THE EDITOR

The maximal kinematical group of the general Schrödinger equation

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Abstract. The maximal kinematical group of the general Schrödinger equation is discussed.

It is well known that the largest group of space–time transformations which leaves covariant the free Schrödinger equation is a 12-parameter Schrödinger group S containing the Galilean transformations (Hagen 1972, Niederer 1972). The conformal dimension of the Schrödinger wavefunction $\phi(x)$ is $-\frac{3}{2}$, so $\phi(x)$ can be interpreted as a probability amplitude (Barut and Xu 1981). In this Letter we shall discuss the covariance of the general Schrödinger equation under group S.

The general Schrödinger equation has the form

$$W_s\phi(x)=0 \tag{1}$$

with the Schrödinger operator

$$W_s=2im\,\partial_t+\nabla^2-2mV \tag{2}$$

where the potential V is Galilean invariant. Equation (1) is covariant under the following space–time transformations

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{v}t+\Lambda\boldsymbol{x}+\boldsymbol{a} \\ t' &= t+\tau \end{aligned} \tag{3}$$

and the wavefunction transforms as

$$\phi'(\boldsymbol{x}'t')=\exp[im(\boldsymbol{v}\cdot\Lambda\boldsymbol{x}+\tfrac{1}{2}v^2t)]\phi(\boldsymbol{x}t). \tag{4}$$

The generators which relate the transformations of equations (3) and (4) are those of the quantum mechanical Galilean group

$$\begin{aligned} P_0 &= \mathrm{i}\,\partial_t \\ \boldsymbol{P} &= -\mathrm{i}\nabla \\ L_{kj} &= -\mathrm{i}(x_k\,\partial_j-x_j\,\partial_k) \\ L_{0j} &= \mathrm{i}(t\,\partial_j-\mathrm{i}mx_j). \end{aligned} \tag{5}$$

Equations (5) together with the generators

$$D = \mathrm{i}(2t\,\partial_t + \boldsymbol{x}\cdot\nabla + \tfrac{3}{2}) \tag{6}$$

$$C = \mathrm{i}(t^2\,\partial_t + t\boldsymbol{x}\cdot\nabla - \tfrac{1}{2}\mathrm{i}m\boldsymbol{x}^2 + \tfrac{3}{2}t) \tag{7}$$

form the Lie algebra of the Schrödinger group (Hagen 1972, Niederer 1972).

The commutations of these two generators of equations (6) and (7) with the Schrödinger operator W_s give

$$[W_s, D] = 2\mathrm{i}(2\mathrm{i}m\,\partial_t + \nabla^2) - [2mV, D] \tag{8}$$

$$[W_s, C] = 2\mathrm{i}t(2\mathrm{i}m\,\partial_t + \nabla^2) - [2mV, C]. \tag{9}$$

Therefore the free Schrödinger equation is covariant with respect to the generators of D and C, whereas the general Schrödinger equation (1) usually is not. The covariance of equation (1) implies

$$[V, D] = 2\mathrm{i}V \tag{10}$$

$$[V, C] = 2\mathrm{i}tV. \tag{11}$$

Insert equations (6) and (7) into (10) and (11) respectively, we have

$$[V, \boldsymbol{x}\cdot\nabla] = 2V \qquad [V, t\,\partial_t] = 0. \tag{12}$$

Equation (12) is the covariance condition of the potential. The general solution of equation (12) has the form

$$V = A_{i_1\ldots i_n}\frac{x_{i_1}\ldots x_{i_n}}{r^{n+2}} \qquad n = 0, 1, 2, \ldots \tag{13}$$

where constant $A_{i_1\ldots i_n}$ is symmetric with respect to the indices $i, \ldots i_n$. Equation (13) gives the general form of the potential V with which equation (1) is covariant under group S.

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